university of groningen

Exam Computer Assisted Problem Solving (CAPS)

April 2nd 2019 14.00-17.00

This exam is 'closed book'. It is NOT allowed to use a book or lecture notes. Only the use of a simple pocket calculator is allowed. Programmable calculators are not allowed, nor the use of electronic devices (tablet, laptop, etc.).

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: [10]

1. The equation $\ln(x) = \dfrac{x}{3}$ has a solution near $x = 1.85$.

   (a) [7] (1) Compute x_1 with Newton's method (one iteration), starting with $x_0 = 2$. Determine the most accurate ('the best') error estimate for x_1.
   (2) Does Newton give even 3rd order convergence for this equation? Explain why!

   (b) [4] Determine a $x_{n+1} = g(x_n)$ method with optimal linear convergence factor for this problem, by introduction and optimisation of a parameter α.

   (c) [7] When $x_{n+1} = e^{x_n/3}$ is used, again with $x_0 = 2$, the first iterations are given by

x_0	x_1	x_2	x_3	x_4
2.00000000	1.94773404	1.91409453	1.89275135	1.87933336

   (1) Will this method converge eventually? Explain why.
   If it does converge, describe the 'speed' of the error reduction.
   (2) Determine an error estimate for x_4.
   (3) Calculate an improved solution by means of Steffensen extrapolation.

   (d) [8] Give a complete program (in pseudo code or Matlab-like language) that solves the intersection problem with an accuracy of tol=1E-6, using the Secant method with optimal efficiency. Use an appropriate stopping criterion.

2. Consider the integral $I = \displaystyle\int_0^1 \frac{e^x}{x+1}\,dx.$

   (a) [7] (1) Explain that the Trapezoidal method will give optimal 2nd order convergence.
   (2) Use the Trapezoidal method on a grid with two segments to approximate I.
   (3) Give an error estimate for the result at (2) using the global error theorem.
   Hint: you may directly use that on [0,1] the function $f''(x)$ has extreme values at $x=0$, $x=1$ and $x \approx 0.6$, without considering $f'''(x)$.

   (b) [4] (1) Apply Simpson's rule on a grid with only one segment [0,1] to approximate I.
   (2) How can one combine the Trapezoidal and Midpoint methods to obtain Simpson?
   Explain why the (global) error estimates cannot be combined in the same way.

   (c) [7] With the Trapezoidal method on finer grids the following results are obtained

   $I(n)$ is the approximation of the integral on a grid with n sub-intervals.

n	$I(n)$
32	1.12544138
64	1.12539991
128	1.12538954
256	1.12538695

   (1) Compute the q-factor. What can you conclude?
   (2) How many segments are required (use powers of 2) for an accuracy of 1.0E-8?
   (3) Compute an improved solution for $I(256)$ through T_2 extrapolation.
   How can one determine the accuracy of $T_2(256)$? Give the required formula(s).

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(c) [8] Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using the Midpoint method (no extrapolation). Use an appropriate error estimate for the stopping criterion.

3. Consider on $[-5\ 0]$ the o.d.e. $y'(x) = (x+1)y(x)$, with boundary condition $y(-5) = 2$.

(a) [7] (1) Use Heun's method (RK2) to compute $y(x)$ at $x = -4.5$ on a grid with $\Delta x = 0.5$.
(2) Use implicit(!) Euler to compute $y(x)$ at $x = -4.5$ on a grid with $\Delta x = 0.5$.
(3) Is implicit Euler always stable in this case? If not, determine the stability limit.

(b) [7] Heun's method RK2 is used on a number of grids ($N = 40, 80, 160, 320$ segments). The table below shows solutions at a selection of x locations.

x_n	$N=40$	$N=80$	$N=160$	$N=320$
-4	7.071417 E-02	6.246023 E-02	6.086396 E-02	6.050695 E-02
-3	6.123759 E-03	5.188492 E-03	5.010061 E-03	4.970097 E-03
-2	1.381633 E-03	1.160728 E-03	1.118601 E-03	1.109151 E-03
-1	8.383685 E-04	7.041135 E-04	6.784912 E-04	6.727405 E-04
0	1.381090 E-03	1.160671 E-03	1.118594 E-03	1.109150 E-03

(1) Is there a stability limit visible? Explain.
Determine the stability limit. Does it confirm your observation?
(2) Give error estimates ϵ for the solutions at $x = -2$ and $x = -1$ on the finest grid.
Why is ϵ at $x = -2$ larger? Does this contradict the effect of error accumulation?
(3) Which N is required (roughly) for an accuracy of 1.0E-8 at $x = -2$?
Is it advisable to use RK2 in that case? Explain.

(c) [4] (1) When the Midpoint method is evaluated over 2 segments, a 2-step explicit method can be derived. Derive this method and give the general formula(s).
(2) Apply this method to compute the solution of the o.d.e. at $x = -4.5$, using a grid-size $\Delta x = 0.25$. Take $y(-4.75) = 0.75$ for the intermediate point.

(d) [8] Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using the Heun method (without extrapolation). Use an appropriate error estimate for the stopping criterion.

4. The bending $w(x)$ (in mm) of a bridge (length $L = 4$ m), due to an uniform load, follows from

$$w''(x) = \alpha\, w(x) + \beta(x - L)x, \quad 0 < x < L$$

with $\alpha = 0.5$ and $\beta = 5.0\text{E-}3$ material constants. Initially, the bridge is clamped at both ends: $w(0) = w(L) = 0$.

(a) [4] (1) Give the matrix-vector system, when the problem is solved on a grid with $N = 2$ segments by means of the matrix method, using the [1 -2 1]-formula for $w''(x)$.
(2) Solve the system to find the bending in the middle of the bridge.

(b) [4] Because of partial damage, the left boundary condition changes: $w'(0) = 0$.
(1) Give the new matrix-vector system ($N = 2$) and compute the value of $w(0)$.
(2) Explain why $w'(0) = 0$ requires a finer grid than the situation with $w(0) = 0$.

(c) [4] The [1 -2 1]-formula for $w''(x)$ can be derived by considering $w''(x) = \frac{d}{dx} w'(x)$ on an equidistantly spaced grid using grid points $x_{[i-1]}$, $x_{[i]}$ and $x_{[i+1]}$, or with Taylor series. Derive a formula for $w''(x)$ needed when a stretched grid with $N = 2$ segments covers the bridge, with $x_{[i+1]} - x_{[i]} = 2\,(x_{[i]} - x_{[i-1]})$.

Total: [100]